Deep learning isochronism

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The isochronism problem

Consider the 1D physical system

$$m\ddot{x} = -\frac{\mathrm{d}U}{\mathrm{d}x}.$$

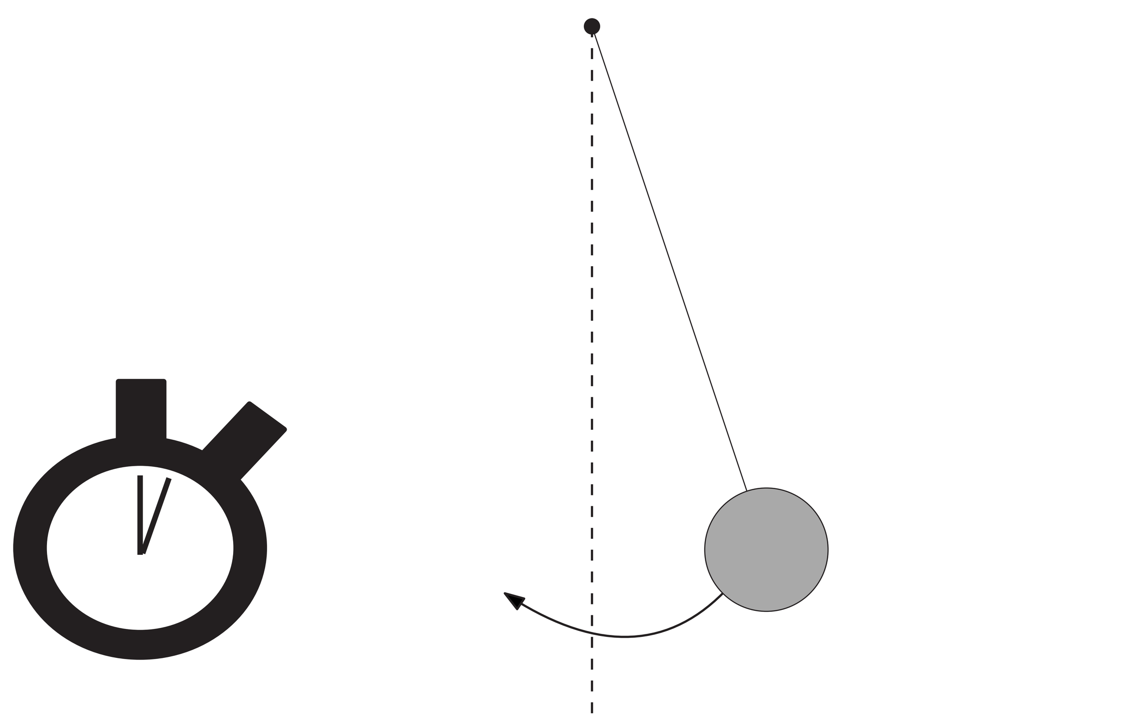
For which U is the system **isochronic** ?

The **harmonic potential** $U(x) = \frac{1}{2}kx^2$ is one solution.
Question : are there other such symmetric functions U ?
Answer : no, necessarily $U(x) \propto x^2$.

Approach

We retrieve this uniqueness result **experimentally** with deep learning.

- Parametrize the unknown force field by a neural network.
- Impose periodicity with a loss on the trajectories.
- Train the neural differential equation.
- Compare the obtained phase portrait with that of the harmonic oscillator.

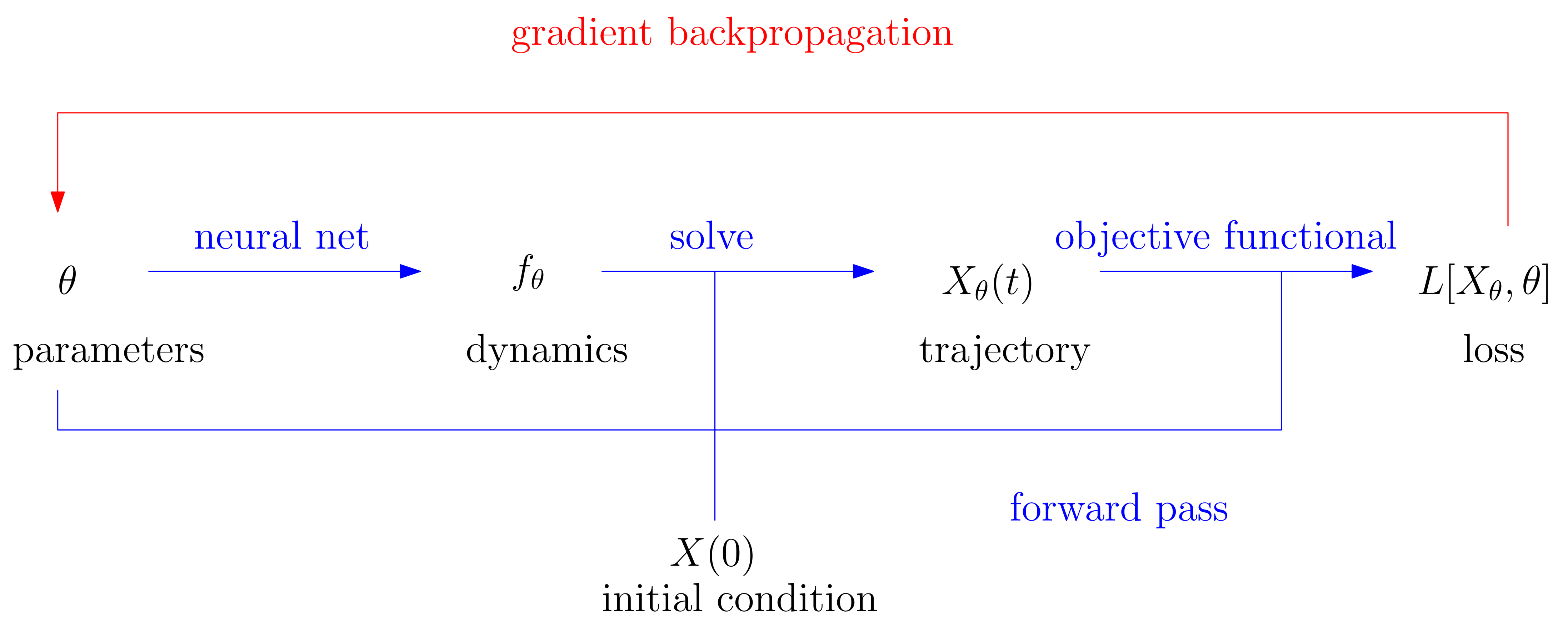


$$m\ddot{x} = \mathrm{NN}_\theta(x)$$

Neural differential equations

Neural differential equations parametrize the flow of an ODE by a neural net : $\frac{\mathrm{d}X}{\mathrm{d}t} = f_\theta(X)$.
Given some loss $L[X_\theta, \theta]$, the gradient is backpropagated through the ODE [Chen *et al.*, 2018].

▹ Learn a flow by imposing a loss on its trajectories.



Architecture of a neural ODE.

Experiment

- Parametrize the dynamics

$$m\ddot{x} = f_\theta(x),$$

where $f_\theta(x)$ is a neural network approximating the force field $f(x) = -\mathrm{d}U/\mathrm{d}x$.

- Measure the **periodicity discrepancy** with the loss

$$L[X_\theta] = \|X_\theta(T) - X_\theta(0)\|^2.$$

- Optimize

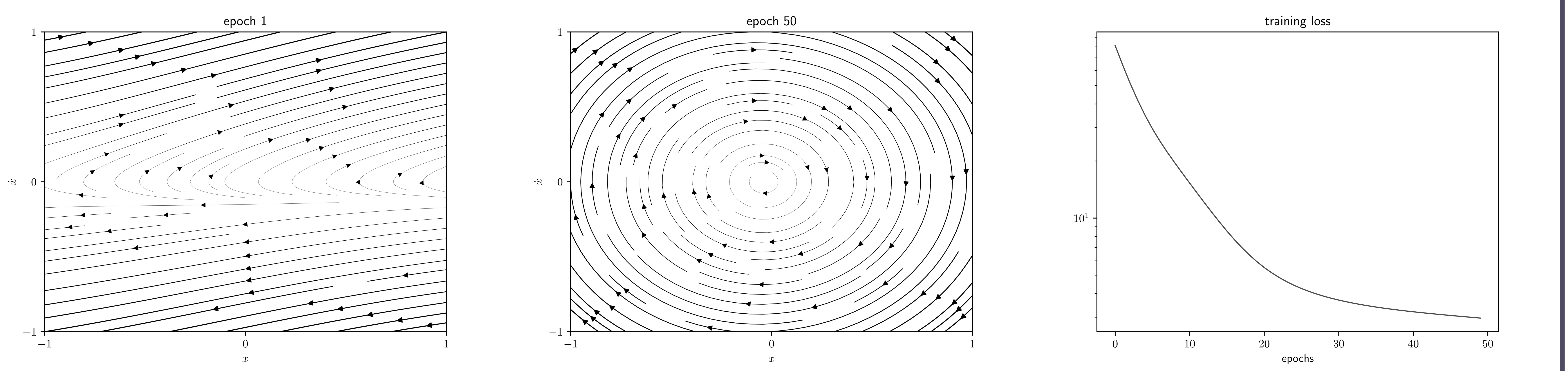
$$\min_\theta L[X_\theta]$$

by gradient descent through the ODE, with random initial conditions

$$X(0) \sim \mathcal{N}(0, I).$$

We trained the loss $N = 100$ training points with $T = 2\pi$. We took for f_θ a 2-layer fully connected architecture with width 16 and tanh nonlinearity.

Results



After training, the trajectories in the phase space are circular, meaning that the obtained neural potential is quadratic : $U(x) = \frac{1}{2}m\omega^2 x^2$, with $\omega = 2\pi/T$.

Theoretical derivation

Inspired from [Osypowski & Olsson, 1986]

- For some energy E, let $y(E)$ denote the amplitude : $-y(E) \leq x \leq y(E)$.
- Write the period in terms of the energy : $T = 2\sqrt{2}\int_0^{y(E)} \frac{\mathrm{d}x}{\sqrt{E - U(x)}}$.
- Changing variables, $T = 2\sqrt{2}\int_0^1 g(uE)\frac{\mathrm{d}u}{\sqrt{u(1-u)}}$ with $g(U) = x'(uE)\sqrt{uE}$.
- This integral is a constant with respect to E, implying that g is a constant, and hence $U(x) \propto x^2$.

Conclusion

We recovered a result from elementary theoretical mechanics using neural differential equations.

▹ The training loss is implicitly defined in terms of the neural net. This is an example of **implicit layer**.

▹ It would be interesting to extend this type of approach to other fields, replacing time translation with other symmetries.